AN ASYMPTOTIC THEOREM FOR MINIMAL SURFACES AND EXISTENCE RESULTS FOR MINIMAL GRAPHS IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. In this paper we prove a general and sharp Asymptotic Theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. As a consequence, we prove that there is no properly immersed minimal surface whose asymptotic boundary Γ_∞ is a Jordan curve homologous to zero in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ such that Γ_∞ is contained in a slab between two horizontal circles of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with width equal to π.

We construct minimal vertical graphs in $\mathbb{H}^2 \times \mathbb{R}$ over certain unbounded admissible domains taking certain prescribed finite boundary data and certain prescribed asymptotic boundary data. Our admissible unbounded domains Ω in $\mathbb{H}^2 \times \{0\}$ are non necessarily convex and non necessarily bounded by convex arcs; each component of its boundary is properly embedded with zero, one or two points on its asymptotic boundary, satisfying a further geometric condition.

1. INTRODUCTION

In this paper we prove an Asymptotic Theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Indeed, we prove a surprising general and sharp nonexistence result. As a consequence, we deduce that there is no complete properly immersed minimal surface whose asymptotic boundary Γ_∞ is a Jordan curve homologous to zero in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ contained in an open slab between two horizontal circles of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with width equal to π. The last statement is still true in a closed slab with width *equal* to π in the class of minimal surfaces continuous up to the asymptotic boundary. This result is sharp in the following sense. We show that for any $\ell > \pi$ there is a Jordan curve $\Gamma_\infty \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ homologous to zero with vertical height equal to ℓ which is the asymptotic boundary of a complete minimal surface, continuous up to its asymptotic boundary Γ_∞. Moreover, this surface is invariant by hyperbolic translations and

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is constituted of two minimal vertical graphs over the exterior of an equidistant curve, symmetric about the horizontal slice $\mathbb{H}^2 \times \{0\}$. In fact, the Jordan curve Γ_∞ is the union of two vertical segments with two half circles in $\partial_\infty \mathbb{H}^2$. Another consequence of our Asymptotic Theorem is that there is no complete properly immersed minimal surface contained in an open slab of width equal to π of $\mathbb{H}^2 \times \mathbb{R}$, such that the vertical projection of its asymptotic boundary on $\partial_\infty \mathbb{H}^2 \times \{0\}$ omits an open arc.

Those results contrast with an analogous situation when the ambient space is the hyperbolic three-space $\mathbb{H}^3$, due to the existence of the minimal vertical graph (taking the upper half-space model) whose asymptotic boundary is any convex curve lying in $\partial_\infty \mathbb{H}^3$. Indeed, the authors have solved the Dirichlet Problem in $\mathbb{H}^3$ for the minimal vertical equation over a convex domain Ω in $\partial_\infty \mathbb{H}^3$, taking any prescribed continuous boundary data on $\partial\Omega$ ([19]). There are also the general results proved by M. Anderson [1] and [2].

We give some geometric conditions to construct minimal vertical graphs in $\mathbb{H}^2 \times \mathbb{R}$ over certain unbounded admissible domains taking certain prescribed finite boundary data and certain prescribed asymptotic boundary data.

To obtain our existence results we establish the *Perron process* when the finite boundary data and the asymptotic boundary data are continuous except maybe at a finite set.

As a consequence, we prove the following. Let Ω be a convex unbounded domain. Let $g : \partial\Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be a bounded function everywhere continuous except at a finite set S. Then g admits an extension u satisfying the minimal vertical equation over Ω such that the total boundary of the graph of u is the union of the graph of g on $(\partial\Omega \cup \partial_\infty \Omega) \setminus S$ with vertical segments at the points of S. This result was obtained independently by M. Rodríguez and H. Rosenberg

We built *barriers* at each convex point of a convex finite boundary, where the boundary data are continuous and bounded and we construct *barriers* at each point of the asymptotic boundary where the asymptotic data is continuous. Our admissible unbounded domains Ω in $\mathbb{H}^2 \times \{0\}$ are non necessarily convex and non necessarily bounded by convex arcs; each component of its boundary is properly embedded with zero, one or two points on its asymptotic boundary, satisfying a further geometric condition: each connected component C_0 of $\partial\Omega$ satisfies the *Exterior circle of (uniform) radius ρ condition*. Particularly, we consider an admissible domain Ω that is the exterior of a C^2 Jordan curve Γ in the horizontal slice.

We obtain the existence of minimal graph M over an admissible domain Ω in $\mathbb{H}^2 \times \{0\}$ such that the finite boundary of M is $\partial\Omega$ and the asymptotic boundary of M is a certain Jordan curve Γ_∞ constituting in the union of bounded continuous vertical graphs with the vertical segments joining the points of discontinuities, such that Γ_∞ is contained inside a certain slab of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ depending on the geometry of Ω.

We consider admissible domains, that we call E-admissible domains, such that each component of the boundary has two points at its asymptotic boundary and has at each point of its finite boundary an exterior equidistant curve. We obtain analogous existence results for E-admissible domains.

2. An asymptotic theorem

In this section we prove an Asymptotic Theorem that ensures some nonexistence results about minimal surfaces with some given asymptotic boundary.

THEOREM 2.1 (Asymptotic Theorem).
  Let $\gamma \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ be an arc. Assume there exist a vertical straight line $L \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ and a subarc $\gamma' \subset \gamma$ such that

  (1) *$\gamma' \cap L \neq \emptyset$ and $\partial\gamma' \cap L = \emptyset$,*
  (2) *γ' stays on one side of L,*
  (3) *$\gamma' \subset \partial_\infty \mathbb{H}^2 \times (t_0, \pi + t_0)$, for some real number t_0.*

Therefore, there is no properly immersed minimal surface (maybe with finite boundary), $M \subset \mathbb{H}^2 \times \mathbb{R}$, with asymptotic boundary γ and such that $M \cup \gamma$ is a continuous surface with boundary.

Proof. By assumption there exists a point p in $\gamma' \cap L$. If there is a vertical segment in $\gamma' \cap L$, we choose p to be the midpoint of this segment. Up to a vertical translation, we can assume that $p \in \partial_\infty \mathbb{H}^2 \times \{0\}$. The vertical projection of γ' on $\partial_\infty \mathbb{H}^2 \times \{0\}$ is an arc β with p as one of the two end points. Let $\varepsilon > 0$ be a real number to be chosen later. Let $q_1, q_2 \in \partial_\infty \mathbb{H}^2 \times \{0\}$ be two distinct points such that $q_1 \in \beta$, $q_2 \notin \beta$ and the Euclidean distance on $\partial_\infty \mathbb{H}^2 \times \{0\}$ from p to q_i is ε, $i = 1, 2$. Let $c \subset \partial_\infty \mathbb{H}^2 \times \{0\}$ be the complete geodesic with asymptotic boundary $\{q_1, q_2\}$ and let $S = c \times \mathbb{R}$ be the vertical geodesic plane defined by c.

Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be a minimal surface (if any) with asymptotic boundary γ and such that $M \cup \gamma$ is a continuous surface with boundary. If ε is small enough we have $S \cap \partial M = \emptyset$. Let $M_0 \subset M$ be the connected component of $M \backslash S$ containing p in its asymptotic boundary. Therefore, the asymptotic boundary of M_0 is a subarc γ_0 of γ' containing p in its

interior: $\gamma_0 \subset \gamma' \subset \gamma$ and $p \in \text{Int}(\gamma_0)$. Let $\beta' \subset \beta \subset \partial_\infty \mathbb{H}^2 \times \{0\}$ be the subarc of β with end points p and q_1. For ε small enough we have $\gamma_0 \subset \beta' \times (-\pi/2, \pi/2)$. By construction there exist two real numbers a and b satisfying $a < 0 < b$, $b - a < \pi$ and $\partial\gamma_0 = \{(q_1, a), (q_1, b)\}$.

Observe that, by continuity, for ε small enough the whole component M_0 is inside the slab $\mathbb{H}^2 \times (-\pi/2, \pi/2)$. Furthermore, the finite boundary ∂M_0 of M_0 is contained in the vertical geodesic plane S. Therefore, there is a complete geodesic $c_1 \subset \mathbb{H}^2 \times \{0\}$ with asymptotic boundary in the open arc $(p, q_2) \subset \partial_\infty \mathbb{H}^2 \times \{0\} \backslash \beta$, such that $M_0 \cap (c_1 \times \mathbb{R}) = \emptyset$.

Let $C \subset \mathbb{H}^2 \times \mathbb{R}$ be a complete catenoid whose a component of the asymptotic boundary stays at height T_1 and the other component at height T_2 such that $T_1 < a < b < T_2$ (such a catenoid exists since $0 < b - a < \pi$), note that $T_2 - T_1 < \pi$, the reader can see the geometric behaviour of the catenoids in Lemma 5.1 or [15]. By continuity, we can choose T_1 and T_2 such that M_0 is entirely contained in the open slab $\mathbb{H}^2 \times (T_1, T_2)$. Finally, let t_1, t_2 be two real numbers satisfying $T_1 < t_1 < a < b < t_2 < T_2$, such that M_0 is entirely contained in the open slab $\mathbb{H}^2 \times (t_1, t_2)$.

Let $\overline{C}$ be the part of C contained in the slab $\{t_1 \leqslant t \leqslant t_2\}$, that is, $\overline{C} = C \cap (\mathbb{H}^2 \times [t_1, t_2])$. Observe that $\overline{C}$ is a compact surface. Up to a hyperbolic translation we can send $\overline{C}$ into the connected component of $\mathbb{H}^2 \times \mathbb{R} \backslash (c_1 \times \mathbb{R})$ not containing p in its asymptotic boundary, so we can assume that $\overline{C}$ has this property.

Let $c_2 \subset \mathbb{H}^2 \times \{0\}$ be a complete geodesic with an asymptotic boundary point in $\text{Int}(\beta')$ and the other asymptotic boundary point in the open arc (p, q_2). We choose c_2 such that c_1 is contained in the component of $(\mathbb{H}^2 \times \{0\}) \backslash c_2$ containing p in its asymptotic boundary. Consider the hyperbolic translations along c_2. Observe that all translated copies of $\overline{C}$ have a component of the finite boundary at height t_1 and the other component at height t_2. Therefore the boundary of any translated copy of $\overline{C}$ has no intersection with $\overline{M_0}$. Consequently some translated copy of $\overline{C}$ must achieve a first interior contact point with M_0, which contradicts the maximum principle. This concludes the proof of the Theorem.

$\square$

COROLLARY 2.1. *Let $\Gamma_\infty \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ be a Jordan curve homologous to zero (in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$). We have the following:*

(1) *Suppose that Γ_∞ is strictly contained in a closed slab between two horizontal circles of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with width equal to π. Then,*

(a) *there is no properly immersed minimal surface M with asymptotic boundary Γ_∞, possibly with finite boundary, such that $M \cup \Gamma_\infty$ is a continuous surface with boundary.*

(b) *there is no complete properly immersed minimal surface with asymptotic boundary Γ_∞ (without any assumption on $M \cup \Gamma_\infty$).*

(2) *Suppose that Γ_∞ is contained in a slab with width equal to π but is not contained in any slab with width strictly less than π. Then, there is no complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$, with asymptotic boundary Γ_∞ such that $M \cup \Gamma_\infty$ is a continuous surface with boundary.*

Proof. The Statement (1a) is a direct consequence of Theorem 2.1. The Statement (1b) is a direct consequence of the proof of Theorem 2.1.

Let us prove the Statement (2). Assume there exists a properly immersed complete minimal surface M with asymptotic boundary $\Gamma_\infty \subset \partial_\infty \mathbb{H}^2 \times [0, \pi]$. By the maximum principle, we deduce that $\mathrm{Int}(M) \subset \mathbb{H}^2 \times (0, \pi)$. We deduce from Theorem 2.1 that Γ_∞ is constitued by two vertical segments of length π: $\{q_1\} \times [0, \pi]$ and $\{q_2\} \times [0, \pi]$, $q_i \in \partial_\infty \mathbb{H}^2$ (identified with $\partial_\infty \mathbb{H}^2 \times \{0\}$), $i = 1, 2$, and two simple arcs $c, \gamma \subset \partial_\infty \mathbb{H}^2 \times [0, \pi]$, the arc c joining the points $(q_1, 0)$ and $(q_2, 0)$ and the arc γ joining the points (q_1, π) and (q_2, π). Therefore we have

$$\Gamma_\infty = (\{q_1\} \times [0, \pi]) \cup \gamma \cup (\{q_2\} \times [0, \pi]) \cup c.$$

Up to an ambiant isometry, we can assume that $q_1 = e^{i\pi/4}$, $q_2 = e^{-i\pi/4}$ and that the vertical projection of Γ_∞ on $\partial_\infty \mathbb{H}^2 \times \{0\}$ is the arc $\{(e^{i\theta}, 0) \mid -\pi/4 \leqslant \theta \leqslant \pi/4\}$.

Let H be the parabolic complete minimal surface (foliated by horocycles) whose asymptotic boundary is the vertical segment $\{-1\} \times [0, \pi] \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with $(\partial_\infty \mathbb{H}^2 \times \{0\}) \cup (\partial_\infty \mathbb{H}^2 \times \{\pi\})$, see [6], [10] and [18]. The "neck" of H is a horocycle N in the slice $\mathbb{H}^2 \times \{\pi/2\}$.

Claim. If $N \cap M = \emptyset$ then $H \cap M = \emptyset$.

Assume by contradiction that $N \cap M = \emptyset$ and $H^+ \cap M \neq \emptyset$, where $H^+ := H \cap (\mathbb{H}^2 \times [\pi/2, \pi])$. For any $\varepsilon > 0$ we denote by H_ε^+ the ε-vertical translated of H^+: $H_\varepsilon^+ = H^+ + \varepsilon \partial/\partial t$. Observe that if $H_\varepsilon^+ \cap M = \emptyset$ for any $\varepsilon > 0$, letting $\varepsilon \to 0$ then M and H^+ would have a first interior point of contact, contradicting the maximum principle. Therefore, there exists $\varepsilon > 0$ such that $H_\varepsilon^+ \cap M \neq \emptyset$ and $(N + \varepsilon \partial/\partial t) \cap M = \emptyset$. Furthermore, the finite and asymptotic boundary of H_ε^+ is far away from $M \cup \partial_\infty M$. Consider the hyperbolic translations along the geodesic β with asymptotic boundary $\{-1, 1\}$, going from 1 to -1.

Thus we would obtain a last interior contact point of M and some translated copy of H_ε^+, which contradicts the maximum principle.

We show in the same way that $N \cap M = \emptyset$ and $H^- \cap M \neq \emptyset$ is not possible. This proves the Claim

Up to a hyperbolic translation along the geodesic β (with asymptotic boundary $\{-1, 1\}$), we can assume that $N \cap M = \emptyset$ (since $-1 \in \partial_\infty \mathbb{H}^2$ is not in the asymptotic boundary of M), and therefore the Claim shows that $H \cap M = \emptyset$. Consider now the translated copies of H, along β, going from -1 to 1. As M is properly immersed, some translated copy of H will have a first contact point with M at a point $p \in M$, which contradicts the maximum principle. This concludes the proof of the Corollary $\square$

The following result is a direct consequence of the proof of the Asymptotic Theorem (Theorem 2.1).

COROLLARY 2.2. *Let $S_\infty \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ be a closed set strictly contained in a slab with width equal to π. Assume that the vertical projection of S_∞ on $\partial_\infty \mathbb{H}^2 \times \{0\}$ omits an open arc. Then, there is no complete properly immersed minimal surface M in $\mathbb{H}^2 \times \mathbb{R}$ with asymptotic boundary S_∞.*

Proof. By assumption there exists a complete geodesic $c \subset \mathbb{H}^2 \times \{0\}$ such that S_∞ is contained in the asymptotic boundary of a component of $\mathbb{H}^2 \times \mathbb{R} \setminus c \times \mathbb{R}$. We call U the other component. Let C be a catenoid, observe that any compact part of C may be mapped into U by an ambient isometry. In this situation, we can proceed as in the proof of Theorem 2.1. $\square$

REMARK 1. We will see in Proposition 2.1 that for any $t_0 > \pi$ there exists a Jordan curve $\Gamma_\infty \subset \partial_\infty \mathbb{H}^2 \times [0, t_0]$, homologous to zero, which is the asymptotic boundary of a properly embedded complete minimal surface. Therefore the results in Theorem 2.1 and Corollary 2.1 are sharp. The formulae of the generated curves in Proposition 2.1 are the same as formulae given by the first author in [18]. The geometric description of the surfaces given in Proposition 2.1 is new. We remark that L. Hauswirth [10] has given a classification of minimal surfaces invariant by hyperbolic translations using another approach.

PROPOSITION 2.1. *Let $q_1, q_2 \in \partial_\infty \mathbb{H}^2$ (identified with $\partial_\infty \mathbb{H}^2 \times \{0\}$), be two distinct asymptotic points. Let $\gamma \subset \mathbb{H}^2$ (identified with $\mathbb{H}^2 \times \{0\}$), be the complete geodesic with asymptotic boundary $\{q_1, q_2\}$. Let us call c_1 (resp. c_2) the closed arc in $\partial_\infty \mathbb{H}^2$ joining q_1 to q_2 (resp. q_2 to q_1) with respect to the counterclockwise orientation.*

There exist a one-parameter family M_d, $d > 0$, of complete properly embedded minimal surfaces, invariant by the hyperbolic translations along γ. The geometric behaviour of M_d is as follows.

(1) *If $d > 1$, then M_d contains the equidistant line γ_d of γ in $\mathbb{H}^2 \times \{0\}$ staying at the distance $\cosh^{-1}(d)$ from γ in the connected component of $\mathbb{H}^2 \backslash \gamma$ whose asymptotic boundary is c_1. Furthermore M_d is symmetric with respect to the slice $\mathbb{H}^2 \times \{0\}$ and we have (see Figure 1(a))*

$$\partial_\infty M_d = (c_1 \times \{-H(d)\}) \cup (c_1 \times \{H(d)\})$$
$$\cup \; (\{q_2\} \times [-H(d), H(d)]) \cup (\{q_1\} \times [-H(d), H(d)]),$$

where

$$H(d) := \int_{\cosh^{-1}(d)}^{+\infty} \frac{d}{\sqrt{\cosh^2 u - d^2}} du, \quad d > 1. \tag{1}$$

Therefore, $\partial_\infty M_d$ is a Jordan curve homologous to zero in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Furthermore, the part $M_d \cap (\mathbb{H}^2 \times [0, H(d)]$ is a graph over the component of $\mathbb{H}^2 \backslash \gamma_d$ whose asymptotic boundary is c_1. Finally, $H(d)$ is a nonincreasing function satisfying

$$\lim_{d \to 1} H(d) = +\infty, \quad \lim_{d \to +\infty} H(d) = \frac{\pi}{2}.$$
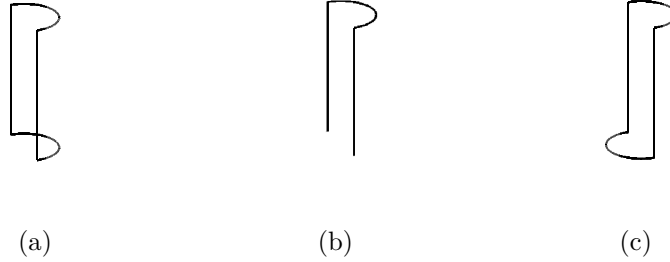

(a) (b) (c)

FIGURE 1.

(2) *If $d = 1$, then M_1 is the surface given by Formula 5 , and its asymptotic boundary is given by (see Figure 1(b))*

$$\partial_\infty M_1 = (\{q_1\} \times (-\infty, 0]) \cup c_1 \cup (\{q_2\} \times (-\infty, 0]).$$

(3) *If $0 < d < 1$, then M_d is an entire vertical graph over $\mathbb{H}^2$ and contains the geodesic $\gamma \times \{0\}$. The asymptotic boundary of M_d*

is given by (see Figure 1(c))

$$\partial_\infty M_d = (c_2 \times \{-G(d)\}) \cup (c_1 \times \{G(d)\})$$
$$\cup \ (\{q_1\} \times [-G(d), G(d)]) \cup (\{q_2\} \times [-G(d), G(d)]),$$

where

$$G(d) := \int_0^{+\infty} \frac{d}{\sqrt{\cosh^2 u - d^2}} du, \quad 0 < d < 1.$$

Therefore, $\partial_\infty M_d$ is a Jordan curve non homologous to zero in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Furthermore $G(d)$ is a nondecreasing function and we have

$$\lim_{d \to 0} G(d) = 0, \quad \lim_{d \to 1} G(d) = +\infty.$$

Proof. We work with the disk model for $\mathbb{H}^2$, so that

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2, \ x^2 + y^2 < 1\}.$$

Therefore the product metric on $\mathbb{H}^2 \times \mathbb{R}$ reads as follows

$$d\tilde{s}^2 = \left(\frac{2}{1 - (x^2 + y^2)} \right)^2 (dx^2 + dy^2) + dt^2,$$

where $(x, y) \in \mathbb{H}^2$ and $t \in \mathbb{R}$.

Up to an isometry, we can assume that $q_1 = -i$ and $q_2 = i$. Therefore we have $c_1 = \{e^{i\theta}; \ -\pi/2 \leqslant \theta \leqslant \pi/2\}$ and $c_2 = \{e^{i\theta}; \ \pi/2 \leqslant \theta \leqslant 3\pi/2\}$.

We consider the following particular geodesic of $\mathbb{H}^2$

$$\Gamma = \{(x, 0), \ x \in (-1, 1) \} \subset \mathbb{H}^2.$$

We can assume that the surfaces invariant under hyperbolic translation along γ (called *hyperbolic surfaces*), are generated by curves in the vertical geodesic plane $P = \Gamma \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$.

On the geodesic Γ we denote by $\rho \in \mathbb{R}$ the signed distance to the origin $(0, 0)$, thus $x = \tanh(\rho/2)$. Therefore the metric on P is

$$ds^2 = d\rho^2 + dt^2.$$

Let us consider a curve in P which is a vertical graph: $c(\rho) = (\rho, \lambda(\rho))$ where λ is a smooth real function defined on a part of $\rho \geqslant 0$. Let us call M the hyperbolic surface generated by c. On M we consider the orientation given by the upward unit normal field. With respect to this orientation the principal curvatures of M are given by

$$k_1(\rho) = \frac{\lambda''}{(1 + \lambda'^2)^{3/2}}(\rho), \quad \text{and} \quad k_2(\rho) = \frac{\lambda'}{\sqrt{1 + \lambda'^2}}(\rho) \tanh(\rho).$$

So that, M is a minimal surface if and only if

$$\lambda'^2 = \frac{d^2}{\cosh^2 \rho - d^2},$$

for some $d \geqslant 0$.

Up to the isometry $(z, t) \rightarrow (z, -t)$, we can assume that λ is a nondecreasing function, that is, $\lambda' \geqslant 0$. Therefore, the condition for M being minimal is

$$\lambda'(\rho) = \frac{d}{\sqrt{\cosh^2 \rho - d^2}} \qquad (2)$$

In the case where $d > 1$ we can choose, up to a vertical translation,

$$\lambda(\rho) = \int_{\cosh^{-1}(d)}^{\rho} \frac{d}{\sqrt{\cosh^2 u - d^2}} du,$$

for $\rho \geqslant \cosh^{-1}(d)$. Setting $v = \cosh u/d - 1$ we obtain:

$$H(d) = \int_0^{+\infty} \frac{dv}{\sqrt{(v+1)^2 - 1}\sqrt{(v+1)^2 - 1/d^2}}.$$

This shows that $H(d)$ is a nonincreasing function. Furthermore

$$\lim_{d \to +\infty} H(d) = \int_0^{+\infty} \frac{dv}{\sqrt{(v+1)^2 - 1}\sqrt{(v+1)^2}},$$
$$= \int_0^1 \frac{dx}{\sqrt{1 - x^2}}, \quad \text{setting } x = \frac{1}{v+1},$$
$$= \frac{\pi}{2}.$$

Claim. We have $\lim_{d \to 1} \lambda(\rho) = +\infty$ for any $\rho > 0$.

This clearly implies that $\lim_{d \to 1} H(d) = +\infty$, since $H(d) = \lim_{\rho \to +\infty} \lambda(\rho)$ and $\lambda(\rho)$ is a nondecreasing function.

Setting $v = \cosh(u)/d - 1$ we get

$$\lambda(\rho) = \int_0^{\cosh(\rho)/d - 1} \frac{1}{\sqrt{v+2}\sqrt{(v+1+\frac{1}{d})(v+1-\frac{1}{d})}} \frac{dv}{\sqrt{v}}$$

$$\geqslant \frac{1}{\sqrt{\frac{\cosh(\rho)}{d} + 1}\sqrt{\frac{\cosh(\rho)+1}{d}}} \int_0^{\cosh(\rho)/d - 1} \frac{dv}{\sqrt{v^2 + (1 - \frac{1}{d})v}}.$$

Denoting by $I(d)$ the last integral, we have

$$I(d) = \frac{2}{1 - \frac{1}{d}} \int_0^{\cosh(\rho)/d - 1} \frac{dv}{\sqrt{(\frac{2v}{1 - \frac{1}{d}} + 1)^2 - 1}}.$$

Setting $s = \frac{2v}{1 - \frac{1}{d}} + 1$ we obtain

$$I(d) = \int_1^{\frac{2\cosh(\rho) - d - 1}{d - 1}} \frac{ds}{\sqrt{s^2 - 1}} = \cosh^{-1}\left(\frac{2\cosh(\rho) - d - 1}{d - 1}\right),$$

from what we deduce that $\lim_{d \to 1} I(d) = +\infty$ for any $\rho > 0$, which concludes the proof of the Claim.

In the case where $d = 1$ we can choose, up to a vertical translation,

$$\lambda(\rho) = \log\left(\frac{e^\rho - 1}{e^\rho + 1}\right),$$

for $\rho > 0$. The hyperbolic surface generated by λ is a vertical graph over the connected component of $\mathbb{H}^2 \backslash \gamma$ whose asymptotic boundary is c_1. This graph takes value $-\infty$ on γ and value zero on c_1 (because $\lim_{\rho \to 0} \lambda(\rho) = -\infty$ and $\lim_{\rho \to +\infty} \lambda(\rho) = 0$). Since this is the unique hyperbolic surface, up to isometry, with unbounded height, we deduce that M_1 is congruent to the hyperbolic surface given by Formula (5).

Finally, in the case where $0 < d < 1$, the function λ is defined for any $\rho \geqslant 0$ and, up to a vertical translation, we can set

$$\lambda(\rho) = \int_0^\rho \frac{d}{\sqrt{\cosh^2 u - d^2}} du.$$

We can extend λ on $\mathbb{R}$ setting $\lambda(\rho) := -\lambda(-\rho)$ for any $\rho \leqslant 0$. Therefore λ is defined on $\mathbb{R}$ and is an odd function, and the hyperbolic surface M_d generated by λ is an entire vertical graph on $\mathbb{H}^2$ symmetric with respect to γ. We can prove in the same way as in the case where $d > 1$ that for any $\rho > 0$ we have $\lim_{d \to 1} \lambda(\rho) = +\infty$. This implies that $\lim_{d \to 1} G(d) = +\infty$.

The other assertions in the Statement are straightforward verifications.

$\square$

3. MINIMAL VERTICAL GRAPHS

There are many notions of graphs in $\mathbb{H}^2 \times \mathbb{R}$, but the notion of *minimal vertical graphs* has appeared in many important theorems. See, for instance [4], [7], [11], [12], [18], [23].

Consider a C^2 function $t = u(x, y)$. The *vertical minimal equation* in $\mathbb{H}^2 \times \mathbb{R}$, is given by the following equation:

$$\text{div}_{\mathbb{H}} \left(\frac{\nabla_{\mathbb{H}} u}{W_u} \right) = 0 \tag{3}$$

where $\text{div}_{\mathbb{H}}$ and $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2$.

Focusing the halfspace model for $\mathbb{H}^2$, with Euclidean coordinates x, y, $y > 0$, the *minimal vertical equation* (3) takes the following form

$$(1 + y^2 u_x^2)u_{yy} + (1 + y^2 u_y^2)u_{xx} - 2y^2 u_x u_y u_{xy} - y u_y(u_x^2 + u_y^2) = 0 \tag{4}$$

There are many explicit examples of entire and complete minimal graphs with nice geometric properties. For instance, in the half-plane model,

(1) The equation [18]

$$t = \ell x, \; x \in (-\infty, \infty), \; y > 0$$

gives rise to an entire minimal graph (left side of Figure 2) symmetric about the geodesic $\{x = 0\}$, that is constant (the constant varying in the interval $(-\infty, \infty)$) on each leaf of the foliation given by geodesics with a fixed common asymptotic boundary point p (in this model $p = \infty$). Thus the asymptotic boundary consists in the union of a vertical line with a complete embedded curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ asymptotic to that line.

(2) The equation [18]

$$t = \frac{\ell}{2} \ln(x^2 + y^2), \qquad y > 0$$

yields an entire minimal graph (right side of Figure 2) symmetric about the geodesic $\{x^2 + y^2 = 1, \; y > 0\}$, that is constant on each leaf of the foliation given by (hyperbolic) translations of a fixed geodesic; hence, the asymptotic boundary consists of two embedded curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with two symmetric ends, each end asymptotic to a half-vertical line.

Of course, the previous examples give two different explicit non trivial minimal graphs over a half-plane of $\mathbb{H}^2$ taking zero boundary value data on a geodesic but having different asymptotic boundaries.

(3) We observe that there exists a function which takes infinite boundary value data on the positive y axis and zero asymptotic value boundary data at the positive x axis (halfspace model for

FIGURE 2. Ball model for $\mathbb{H}^2 \times \{0\}$

$\mathbb{H}^2$), invariant by *hyperbolic translations* [18].

$$t = \ln\left(\frac{\sqrt{x^2 + y^2} + y}{x}\right), \qquad y > 0, x > 0 \tag{5}$$

Notice that (5) yields a complete vertical minimal graph over a domain bounded by a geodesic in $\mathbb{H}^2 \times \{0\}$, taking infinite boundary value data on the geodesic and zero asymptotic boundary value data on an arc L of $\partial_\infty \mathbb{H}^2 \times \{0\}$. The asymptotic boundary of the graph is then the union of L with the two upper half vertical lines arising from the end points of L. We will use this special minimal vertical graph as a *barrier* at an asymptotic boundary point.

We remark that the miniaml surface given by Formula (5) was used by P. Collin and H. Rosenberg [4] in the important construction of entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ that are conformally the complex plane $\mathbb{C}$, disproving a conjecture by R. Schoen.

In the Poincaré disk model for $\mathbb{H}^2$ with Euclidean coordinates x, y, $x^2 + y^2 < 1$, the *minimal vertical equation* (3) becomes

$$\mathcal{D}(u) := (1 + \frac{(1 - x^2 - y^2)^2}{4} u_x^2)u_{yy} + (1 + \frac{(1 - x^2 - y^2)^2}{4} u_y^2)u_{xx}$$

$$-2\frac{(1 - x^2 - y^2)^2}{4} u_x u_y u_{xy} + 2\frac{(1 - x^2 - y^2)}{4}(xu_x + yu_y)(u_x^2 + u_y^2) = 0 \tag{6}$$

We observe that equation (6) is a second order quasilinear strictly elliptic equation for all *real values* of the independent variables x, y. Moreover, the eigenvalue of the associate matrix are 1 and $W_u = 1 + \frac{(1 - x^2 - y^2)^2}{4}(u_x^2 + u_y^2)$. The same observation holds for the

equation (4), replacing $\frac{(1-x^2-y^2)^2}{4}$ by y^2. Hence we conclude that both equations are regular and strictly (uniformly) elliptic up to the asymptotic boundary of $\mathbb{H}^2 \times \{0\}$. For this reason we can state the classical maximum principle and uniqueness for prescribed continuous finite and asymptotic boundary data.

THEOREM 3.1 (Classical maximum principle). *Let $g_1, g_2 : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be continuous functions satisfying $g_1 \leqslant g_2$. Let $u_i : \overline{\Omega} \to \mathbb{R}$ be a continuous extension of g_i satisfying the minimal equation (6) on Ω, $i = 1, 2$. Then $u_1 \leqslant u_2$.*

Proof. The proof is classical elliptic theory, since the minimal equation (6) is strictly elliptic up to the asymptotic boundary. A geometric approach can be done in this way. Assume that $u_1(p) > u_2(p)$ at some point $p \in \Omega$. Then, lifting the graph of u_2 vertically we obtain a last interior contact point between the graph of u_1 and the graph of u_2, which gives a contradiction by the interior maximum principle. $\qquad\square$

We will solve some Dirichlet problems over certain unbounded domains, given certain prescribed finite boundary data and given certain prescribed asymptotic boundary data.

Among such domains we will consider exterior domains Ω. Of course, the classical examples of such minimal graphs over an exterior domain are given by the one parameter family of half-catenoids, see Lemma 5.1. We will use this family as *barriers*. We show some generating curves in Figure 3, where $R = \tanh \rho/2$, and ρ is the hyperbolic distance from the axe t.

We remark that we use also as *barriers* the one-parameter family of minimal surfaces invariant by hyperbolic translations given by Proposition 2.1.

4. THE PERRON PROCESS FOR THE MINIMAL VERTICAL EQUATION

In the product $\mathbb{H}^2 \times \mathbb{R}$, we consider the disk model for the hyperbolic plane $\mathbb{H}^2$. Let $\Omega \subset \mathbb{H}^2 \times \{0\}$, be a domain. In $\overline{\mathbb{H}^2} \times \{0\}$, we have that $\partial\overline{\Omega} = \partial\Omega \cup \partial_\infty\Omega$, where $\partial\Omega \subset \mathbb{H}^2 \times \{0\}$ and $\partial_\infty\Omega \subset \partial_\infty\mathbb{H}^2 \times \{0\}$.

DEFINITION 1 (Problem (P)). Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function except maybe at a finite set S of points (discontinuities). We consider the Dirichlet problem, say Problem (P), for the minimal vertical equation (6) taking at any point of $\partial\Omega \cup \partial_\infty\Omega \setminus S$, prescribed boundary (finite and asymptotic) value data g.

Let $u : \overline{\Omega}_S := \overline{\Omega} \setminus S \to \mathbb{R}$ be a continuous function.
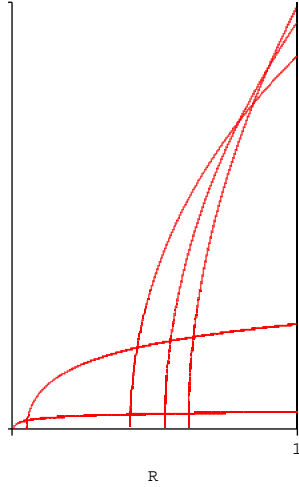
FIGURE 3.   Ball model for $\mathbb{H}^2 \times \{0\}$

Let $U \subset \Omega$ be a closed round disk in $\mathbb{H}^2 \times \{0\}$. If $u_{|\partial U}$ is a C^1 function then solving the Plateau problem [13] and using a standard adaptation of Rado's Theorem [17] (since $u_{|\partial U}$ is a vertical graph over a circle), it follows that $u_{|\partial U}$ has an unique minimal extension $\tilde{u}$ on U, continuous up to ∂U. If $u_{|\partial U}$ is C^0, one uses an approximation argument or uses a local barrier at a boundary point of U. We then define the continuous function $M_U(u)$ on $\overline{\Omega}_S$ by:

$$M_U(u)(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega}_S \setminus U \\ \tilde{u}(x) & \text{if } x \in U \end{cases} \tag{7}$$

We say that u is a *subsolution* (resp. *supersolution*) of (P) if:

i) For any closed round disk $U \subset \Omega$ we have
$u \leqslant M_U(u)$ (resp. $u \geqslant M_U(u)$).

ii) $u\,|_{\partial\Omega \cup \partial_\infty\Omega} \leqslant g$ (resp. $u\,|_{\partial\Omega \cup \partial_\infty\Omega} \geqslant g$).

REMARK 2. We now give some classical facts about subsolutions and supersolutions, see [5], [19].

(1) It is easily seen that if u is C^2 on Ω, the condition i) above is equivalent to $\mathcal{D}u \geqslant 0$ for subsolution or $\mathcal{D}u \leqslant 0$ for supersolution.

(2) As usual if u and v are two subsolutions (resp. supersolutions) of (P) then $\sup(u, v)$ (resp. $\inf(u, v)$) again is a subsolution (resp. supersolution).

(3) Also if u is a subsolution (resp. supersolution) and $U \subset \Omega$ is a closed round disk then $M_U(u)$ is again a subsolution (resp. supersolution).

(4) Let ϕ (resp. u) be a supersolution (resp. a subsolution) of Problem (P) such that $u \leqslant \phi$, then we have $M_U(u) \leqslant M_U(\phi) \leqslant \phi$ for any disk U with $\overline{U} \subset \Omega$.

Note that if ϕ and u are continuous on $\overline{\Omega}$ then necessarily $u \leqslant \phi$ on Ω.

Note also that due to the nature of Equation (6), Ω is a bounded domain in $\overline{\mathbb{H}^2} \times \{0\}$.

DEFINITION 2 (Barriers). We consider the Dirichlet Problem (P), see Definition 1. Let $p \in \partial\Omega \cup \partial_\infty\Omega$, be a boundary point where g is continuous.

(1) Suppose that for any $M > 0$ and for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_k$ of p in $\mathbb{R}^2$ and a function ω_k^+ (resp. ω_k^-) in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that

  i) $\omega_k^+(x) \mid_{(\partial\Omega \cup \partial_\infty\Omega) \cap \mathcal{N}_k} \geqslant g(x)$ and $\omega_k^+(x) \mid_{\partial\mathcal{N}_k \cap \Omega} \geqslant M$
  (resp. $\omega_k^-(x) \mid_{(\partial\Omega \cup \partial_\infty\Omega) \cap \mathcal{N}_k} \leqslant g(x)$ and $\omega_k^-(x) \mid_{\partial\mathcal{N}_k \cap \Omega} \leqslant -M$)

  ii) $\mathcal{D}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{D}(\omega_k^-) \geqslant 0$) in $\mathcal{N}_k \cap \Omega$,

  iii) $\lim_{k \to +\infty} \omega_k^+(p) = g(p)$ (resp. $\lim_{k \to +\infty} \omega_k^-(p) = g(p)$).

(2) Suppose that there exists a supersolution ϕ (resp. a subsolution η) in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi(p) = g(p)$ (resp. $\varphi(p) = g(p)$).

In both cases (1) or (2) we say that p admits a *superior barrier* $(\omega_k^+, k \in \mathbb{N}$ or $\phi)$ (resp. *inferior barrier* $\omega_k^-, k \in \mathbb{N}$ or $\varphi)$ for the Problem (P). If p admits a superior and an inferior barrier we say more shortly that p admits a *barrier*.

EXAMPLE 4.1. [Barrier at any convex point for any bounded continuous boundary data g]. The construction of B. Nelli and H. Rosenberg, the Scherk type minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ over a geodesic triangle, taking zero values data on two sides and infinite at the other side [14] is given in Example 4.3. The geodesic triangle and the boundary data are drawn in Figure 4.

We consider now these Scherk type surfaces when the geodesic triangle Δ is isosceles and the zero value data are taken on the two sides with equal length and $-\infty$ on the other side. We show that these surfaces can be used as a upper barrier (in the sense of Definition 2-(1)) at any convex point $p_0 \in \partial\Omega$, for any boundary bounded data g continuous at p_0. For the lower barrier the construction is analogous.

Let Δ be a geodesic isosceles triangle in $\mathbb{H}^2 \times \{0\}$ with sides A, C_1 and C_2, with $|C_1| = |C_2|$. Let ω be the solution of the minimal equation
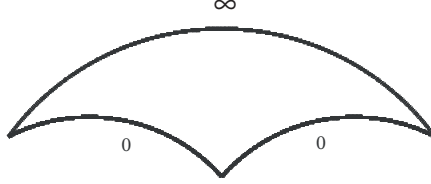
FIGURE 4.

taking zero value data on C_1 and C_2, and $-\infty$ on A. Let S be the graph of ω. Let a be the common vertex of C_1 and C_2. Let γ be the axe of symmetry of Δ, hence $a \in \gamma$. Let $\{b\} = \gamma \cap A$. Let β be a geodesic intersecting Δ orthogonal to γ at a point $d \in (a, b)$. Set $\beta \cap C_1 = \{c\}$.

We claim the following:

(1) ω along γ is nonincreasing in $[a, b]$.
(2) ω along γ is nonincreasing in $[c, d]$.

Assume momentarily the Claim. Let $p_0 \in \partial\Omega$ be a convex point and let g be a boundary data continuous at p_0. Let $M > 0$ be any positive real number. It suffices to show that for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_k$ of p_0 in $\mathbb{R}^2$ and a function ω_k^+ in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that

i) $\omega_k^+(x) \mid_{\partial\Omega \cap \mathcal{N}_k} \geqslant g(x)$ and $\omega_k^+(x) \mid_{\partial\mathcal{N}_k \cap \Omega} \geqslant M$

ii) $\mathcal{D}(\omega_k^+) = 0$ in $\mathcal{N}_k \cap \Omega$,
iii) $\omega_k^+(p_0) = g(p_0) + 1/k$.

By continuity there exists $\epsilon > 0$ such that for any $p \in \partial\Omega$ such that $\mathrm{dist}(p, p_0) < \epsilon$ we have $g(p) < g(p_0) + 1/k$. By assumption there exists an open geodesic arc $\gamma^\perp$, through p_0 such $\gamma^\perp \cap \Omega = \emptyset$. We may assume that the disk $D_\epsilon(p_0)$ intersects $\gamma^\perp$ at two points.

We choose Δ such that $p_0 \in \gamma$, $|A| < \epsilon$, $\overline{\Omega} \cap A = \emptyset$, and γ orthogonal to $\gamma^\perp$ at p_0. Let $M_1 > \max\{M, g(p_0) + 1/k\}$. We consider the Scherk surface (graph of ω) taking M_1 boundary value data on C_1, C_2 and

$-\infty$ on A. By continuity, there exists a point p_1 at γ where $\omega(p_1) = g(p_0) + 1/k$. Up to a horizontal translation along γ sending p_1 to p_0, we may assume that $\omega(p_0) = g(p_0) + 1/k$. Therefore we set $\mathcal{N}_k = \Delta \cap \Omega$ and $\omega_k^+ = \omega \mid_{\mathcal{N}_k}$ is the restriction of ω to $\mathcal{N}_k$. The Claim shows that $\omega_k^+(x) \mid_{\partial\Omega \cap \mathcal{N}_k} \geqslant g(x)$, as desired.

   We will now proceed the proof of the Claim. Let $p_1, p_2 \in [a, b]$ such that $p_1 < p_2$. Let $p_3 \in (p_1, p_2)$ be the middle point in the segment $[p_1, p_2]$ and let γ_3 be the geodesic orthogonal to γ at p_3. Let S_3 be the connected component of $S \setminus (\gamma_3 \times \mathbb{R})$ containing $(a, 0)$. Now maximum principle shows that symmetric of S_3 with respect to $\gamma_3 \times \mathbb{R}$ is above S, since this is true on the boundary. Hence $\omega(p_1) > \omega(p_2)$, as desired. The proof of the second part of the Claim is analogous, considering the reflections about the vertical geodesic planes orthogonal to $[c, d]$. The same argument also shows that S is symmetric about the vertical geodesic plane $\gamma \times \mathbb{R}$. This accomplishes the construction of the desired barrier.

EXAMPLE 4.2 (Barrier at an asymptotic point). The surface given by Formula (5), may be seen as a complete vertical minimal graph over a domain bounded by a geodesic in $\mathbb{H}^2 \times \{0\}$, taking infinite boundary value data on the geodesic and zero asymptotic boundary value data on an arc L of $\partial_\infty \mathbb{H}^2 \times \{0\}$. The asymptotic boundary of the graph is then the union of L with the two half vertical lines arising from the end points of L, see Figure 1(b). We can therefore choose the geodesic as small as we wish in the Euclidean sense, because the minimal equation extends smoothly to $\overline{\mathbb{H}^2} \times \{0\}$. Then we can put a copy of it above and bellow the graph of g at any point p where g is continuous. Thus we obtain a barrier at any point p of $\partial_\infty \Omega$ where g is continuous, in the sense of Definition 2-(1).

THEOREM 4.1 (Perron process). *Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function except, maybe, at a finite set S. Suppose that the Dirichlet Problem (P) has a supersolution ϕ. Set $\mathcal{S}_\phi = \{\varphi, \text{ subsolution of } (P), \varphi \leqslant \phi\}$. Assume that $\mathcal{S}_\phi \neq \emptyset$. We define for each $x \in \overline{\Omega} \setminus S$*

$$u(x) = \sup_{\varphi \in \mathcal{S}_\phi} \varphi(x).$$

We have the following:

   (1) *The function u is C^2 on Ω and satisfies the minimal equation (6).*
   (2) *Let $p \in \partial\Omega$ be a finite boundary point where g is continuous. Suppose that p admits a* barrier *in the sense of Definition 2−(1).*

Then the solution u is continuous at p and satisfies $u(p) = g(p)$. In particular, if $\partial\Omega$ is convex at p then u extends continuously at p and $u(p) = g(p)$.

(3) *Let $p \in \partial_\infty\Omega$ be an asymptotic boundary point where g is continuous. Then p admits a barrier, u is continuous at p and satisfies $u(p) = g(p)$; that is, if (x_n) is a sequence in $\mathbb{H}^2 \times \{0\}$ such that $x_n \to p$ in the Euclidean sense then $u(x_n) \to g(p)$. Particularly, if g is continuous on $\partial_\infty\Omega$ then the asymptotic boundary of the graph of u is the restriction of the graph of g to $\partial_\infty\Omega$.*

(4) *Let $q \in \partial_\infty\mathbb{H}^2$ be an interior point of $\partial_\infty\Omega$ where g is discontinuous. Then the vertical segment*
$$\{(q,t),\ t \in [A := \liminf_{x \to q,\ x \neq q} g(x),\ B := \limsup_{x \to q,\ x \neq q} g(x)],\ x \in \partial\Omega \cup \partial_\infty\Omega\}$$
belongs to the asymptotic boundary of the graph of u. In particular, if $A = -\infty$ and $B = +\infty$, then the whole vertical line $\{q\} \times \mathbb{R}$ belongs to the asymptotic boundary.

Proof. Observe that for any $\varphi \in \mathcal{S}_\phi$, $M_U(\varphi) \in \mathcal{S}_\phi$, for any closed disk $U \subset \Omega$. Observe also that the basic *compactness theorem* holds for the minimal vertical equation, see [8], [22], [23] and [12]. The proof of Statements (1) and (2) follows from classical arguments as in Theorem 3.4 in [19], see also the classical reference [5]. The last assertion of Statement (2) follows from Example 4.1.

The Statement (3) follows from the previous construction of a suitable barrier, in the sense of Definition 2-(1), at any point p of $\partial_\infty\Omega$ where g is continuous, see Example 4.2.

The proof of Statement (4) follows from a continuity argument. Indeed, as g is discontinuous at q we have $A \neq B$. let $t_0 \in (A, B)$. Let (x_n) and (y_n), $n \in \mathbb{N}$, be two sequences in $\partial_\infty\Omega$, such that $x_n, y_n \to q$, $\lim g(x_n) = A$, and $g(y_n) = B$. We can assume that $g(x_n) < t_0 < g(y_n)$, for any n. Let Γ_n be a closed arc joining in $\overline{\Omega}$ the point x_n to y_n, close to q in the Euclidean sense and such that $\Gamma_n \cap \partial_\infty\Omega = \{x_n, y_n\}$. Notice that the restriction of the graph of u to the closed arc Γ_n is continuous and intersects the slice $\mathbb{H}^2 \times \{t_0\}$, at some point (z_n, t_0), where z_n is an interior point of Γ_n, and $z_n \to q$ as $n \to \infty$. Hence, (q, t_0) belongs to the asymptotic boundary of the graph of u, for any $t_0 \in [A, B]$ (as the asymptotic boundary is a closed set). This completes the proof of the Theorem.

$\square$

COROLLARY 4.1. *Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function everywhere continuous except maybe at a finite set $S \subset \partial\Omega \cup \partial_\infty\Omega$. Assume that the finite boundary*

$\partial\Omega$ *is convex or, alternatively, that each finite boundary point admits a barrier.*

Then, g admits an extension $u : \overline{\Omega} \setminus S \to \mathbb{R}$ satisfying the minimal vertical equation (6). Furthermore, the total boundary of the graph of u (that is the finite and asymptotic boundary) is the union of the graph of g on $(\partial\Omega \cup \partial_\infty\Omega) \setminus S$ with the vertical segments

$$\{(q,t), \ t \in [A := \liminf_{x \to q, \ x \neq q} g(x), \ B := \limsup_{x \to q, \ x \neq q} g(x)], \ x \in \partial\Omega \cup \ \partial_\infty\Omega\}$$

at any $q \in S$.

Proof. Since g is bounded, there are some constant functions which are supersolutions and other which are subsolutions of Problem (P). We consider a slight variation of Perron process taking the set $\mathcal{S}$ of continuous subsolutions of (P). Let u be the solution given by the Perron process (Theorem 4.1). It follows from Theorem 4.1 that the total boundary of the graph of u contains the union of the graph of g on $(\partial\Omega \cup \partial_\infty\Omega) \setminus S$ with the vertical segments given in the Statement at any $q \in S \cap \partial_\infty\Omega$. If $q \in S$ is on $\partial\Omega$ or is not an interior point of $\partial_\infty\Omega$ then, taking into account that each finite boundary point has a barrier by assumption, we can prove in the same way that the vertical segment $[A, B]$ is contained in the total boundary of the graph of u.

For any $q_i \in S$ we set $A_i := \liminf_{x \to q_i, \ x \neq q_i} g(x)$ and $B_i := \limsup_{x \to q_i, \ x \neq q_i} g(x)$, $x \in \partial\Omega \cup \ \partial_\infty\Omega$.

It remains to show that for any $q_i \in S$ and any real number t satisfying $t > B_i$ or $t < A_i$ the point (q_i, t) is not in the total boundary of the graph of u.

Assume first that $t > B_i$. Let $\varepsilon > 0$ be a real number satisfying $B_i + \varepsilon < t$. There exists a continuous function $g^+ : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ such that $g^+ > g$ on $(\partial\Omega \cup \partial_\infty\Omega) \setminus S$ and $g^+(q_j) = q_j + \varepsilon$ for any $q_j \in S$. Then the minimal extension u^+ of g^+ given by the Perron process is continuous up to $\overline{\Omega}$. It follows that u^+ is a supersolution of Problem (P) for the boundary data g and, consequently we have $\varphi \leqslant u^+$ on $\overline{\Omega}$ for any $\varphi \in \mathcal{S}$. It follows that the point (q_i, t) is not in the total boundary of the graph of u.

Assume now that $t < A_i$ and consider a continuous function $g^- : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ such that $g^- < g$ on $(\partial\Omega \cup \partial_\infty\Omega) \setminus S$ and $g^-(q_j) = q_j - \varepsilon$ for any $q_j \in S$. Since the minimal extension of g^- is a subsolution of Problem (P), we infer that the point (q_i, t) is not in the total boundary of the graph of u. This concludes the proof of the Corollary.

$\square$

REMARK 3.

(1) It follows from Corollary 4.1 that if Ω is a convex unbounded domain, then there exists an unique minimal vertical graph over Ω taking any prescribed bounded continuous finite and asymptotic boundary data.

(2) In the special case when $\Omega = \mathbb{H}^2$, consider a bounded function g on $\partial_\infty \mathbb{H}^2 \times \{0\}$, continuous except maybe at a finite set of points S. With the aid of Corollary 4.1 we see that g admits a minimal entire extension u. If g is continuous, we remark that uniqueness of the extension follows from Theorem 3.1.

This problem when g is continuous on $\partial_\infty \mathbb{H}^2$, is called *Dirichlet problem at infinity* and was solved by B. Nelli and H. Rosenberg [14].

EXAMPLE 4.3. Let Δ be a geodesic triangle in $\mathbb{H}^2 \times \{0\}$ with sides A, C_1 and C_2. We want to show that there exists a minimal Scherk type graph over Δ taking zero boundary value data on the interior of $C_1 \cup C_2$ and taking $+\infty$ as boundary value data on A. This is proved by B. Nelli and H. Rosenberg in [14].

For this purpose we first show that for any $n \in \mathbb{N}$ there exists a solution u_n of the minimal equation on the interior of Δ taking zero boundary value data on the interior of $C_1 \cup C_2$ and taking n as boundary value data on A. We consider the set $\mathcal{S}_n$ of continuous functions φ on Δ satisfying:

(1) For any closed round disk $U \subset \text{int}\,\Delta, \varphi \leqslant M_U(\varphi)$, where $M_U(\varphi)$ is given in Formula (7)

(2) $\varphi \leqslant 0$ on the interior of $C_1 \cup C_2$

(3) $\varphi \leqslant n$ on A.

For any subarc C' of $C_1 \cup C_2$ and any subarc A' of A there is continuous subsolutions and supersolutions on Δ assuming zero boundary value data on C' and n boundary value data on A'. Those functions give barriers at any interior point of the sides A, C_1 and C_2. Therefore the solution u_n given by the Perron process, Theorem 4.1-(1), assumes the desired boundary value data

Let A_∞ be the complete geodesic containing A. Taking into account Formula (5), let ϕ be the minimal graph over the half-plane with boundary A_∞ that contains Δ, taking $+\infty$ as boundary value data on A_∞ and zero asymptotic boundary value data. We will write down a slight variation of Perron process.

Let $\mathcal{S}_\phi$ be the family of continuous functions φ defined on $\text{int}\,\Delta \cup \text{int}(C_1 \cup C_2)$ satisfying:

(1) For any closed round disk $U \subset \text{int}\,\Delta, \varphi \leqslant M_U(\varphi)$, where $M_U(\varphi)$ is given in Formula (7)

(2) $\varphi \leqslant 0$ on the interior of $C_1 \cup C_2$
(3) $\varphi \leqslant \phi$

Notice that the functions u_n construted above belong to $\mathcal{S}_\phi$. Therefore we infer that the solution u given by Perron process assumes infinite boundary value data on A. We claim that u takes zero boundary value data on the interior of $C_1 \cup C_2$. Actually, let C_3 be an arc of geodesic lying in Δ joining a point c_1 on C_1 to a point c_2 on C_2. Let $a = C_1 \cap C_2$ and let Δ_0 be the geodesic triangle with vertices a, c_1 and c_2.

Let f be the restriction of ϕ to C_3. Notice that the solution of the Dirichlet problem on Δ_0 taking zero boundary value data on the sides $[a, c_1], [a, c_2]$ and f on the side $[c_1, c_2]$ gives rise to a superior barrier at any point of the interior of $C_1 \cup C_2$. Of course the zero function is an inferior barrier to the problem. Thus u takes the desired boundary value data, as we claimed.

EXAMPLE 4.4. We consider a geodesic triangle Δ in $\mathbb{H}^2 \times \{0\}$ with two vertices a, b on $\partial_\infty \mathbb{H}^2 \times \{0\}$ and a third vertex c on $\mathbb{H}^2 \times \{0\}$. Doing a similar construction as in Example 4.3, we can solve our Dirichlet problem on Δ taking infinite boundary value data on the complete geodesic (a, b) and zero boundary value data on the two other sides.

Assume now that the interior angle at vertex c is π/k, $k \in \mathbb{N}^*$. Using Schwarz reflection on the geodesics arcs (c, a), (c, b), and successively about the geodesic boundaries as well, we obtain a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. Since the angle at the vertex c is π/k, $k \in \mathbb{N}^*$, we get a complete graph over an ideal geodesic polygon with $2k$ sides, taking successively boundary values $+\infty$ and $-\infty$. These minimal complete graphs can also be built combining some results on harmonic maps from the complex plane into the hyperbolic plane, done in [24], [9] and [11]. We observe that these examples are a particular case of a general result found in [4].

5. MINIMAL GRAPHS WITH FINITE AND ASYMPTOTIC BOUNDARY IN $\mathbb{H}^2 \times \mathbb{R}$

LEMMA 5.1. *Let $\rho > 0$ and let $\mathcal{C}_\rho \subset \mathbb{H}^2 \times \{0\}$ be a circle of radius ρ. Then there exists a unique catenoid $\mathcal{M}_\rho$ in $\mathbb{H}^2 \times \mathbb{R}$ orthogonal to the slice $\mathbb{H}^2 \times \{0\}$ along $\mathcal{C}_\rho$. Its asymptotic boundary is $\partial_\infty \mathbb{H}^2 \times \{\pm t_0\}$, for some $0 < t_0$, where $t_0 := f(\rho)$ is an increasing function of ρ given by*

$$f(\rho) = \int\limits_{\rho}^{\infty} \frac{\sinh \rho}{\sqrt{\sinh^2 r - \sinh^2 \rho}} \, \mathrm{d}r \qquad (8)$$

Furthermore, $\lim\limits_{\rho \to 0} f(\rho) = 0$, and $\lim\limits_{\rho \to \infty} f(\rho) = \pi/2$.

The proof of Lemma 5.1 follows from Proposition 5.1 of [15] and [20]. For later use we call $\mathcal{M}_\rho^+$ (resp. $\mathcal{M}_\rho^-$) the part of the catenoid $\mathcal{M}_\rho$ in $\mathbb{H}^2 \times [0, \infty)$(resp. in $\mathbb{H}^2 \times (-\infty, 0]$).

PROPOSITION 5.1 (A characterization of minimal vertical graphs). *Let M be a minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, whose finite boundary is a Jordan curve Γ and whose asymptotic boundary is $\partial_\infty \mathbb{H}^2 \times \{t_0\} \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$, $t_0 \geqslant 0$. Assume that Γ is a vertical graph over a Jordan curve $C \subset \mathbb{H}^2 \times \{0\}$. Assume also that the vertical projection of M is contained in $\mathrm{ext}\, C$.*
Then M is a vertical graph. Furthermore, if $\Gamma = C$, then $t_0 < \pi/2$ and M inherits all symmetries of Γ. Particularly, if Γ is an horizontal circle then M is part of a catenoid.

Proof. The proof is somewhat straightforward. We will just sketch it as follows. The first statement is a consequence of Alexandrov Reflection Principle on horizontal slices doing vertical reflections. The second statement ($t_0 < \pi/2$) is a consequence of Lemma 5.1 using the family of catenoids, coming from the infinity towards M. The third statement is a consequence of Alexandrov reflection Principle on vertical geodesic planes.

$\square$

The following Remark is inferred from Lemma 5.1 and maximum principle.

REMARK 4. Let C_ρ be a circle of radius ρ in $\mathbb{H}^2 \times \{0\}$, and let $\Gamma_\infty \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$, be a Jordan curve that is a vertical graph over $\partial_\infty \mathbb{H}^2 \times \{0\}$. If the height function t of Γ_∞ satisfies $t > f(\rho)$ there is no minimal vertical graph over $\mathrm{ext}(C_\rho)$ whose finite boundary is C_ρ and whose asymptotic boundary is Γ_∞.

DEFINITION 3 (Admissible unbounded domains in $\mathbb{H}^2$). Let Ω be an unbounded domain in the slice $\mathbb{H}^2 \times \{0\}$ and let $\partial\Omega$ be its boundary. We say that Ω is an *admissible domain* if each connected component C_0 of $\partial\Omega$ satisfies one of the following conditions:

(1) C_0 is a Jordan curve.

(2) C_0 is a properly embedded curve such that the asymptotic boundary is one point.

(3) C_0 is a properly embedded curve such that the asymptotic boundary is two distinct points.

Finally, each connected component C_0 of $\partial\Omega$ satisfies the *Exterior circle of (uniform) radius ρ condition*, that is, at any point $p \in C_0$ there exists a circle C_ρ of radius ρ such that $p \in C_0 \cap C_\rho$ and $\overline{\text{int } C_\rho} \cap \Omega = \emptyset$.

If Ω is an unbounded admissible domain then we denote by ρ_Ω the supremum of the set of these ρ.

If the components of $\partial\Omega$ are compact, we set $C := \partial\Omega$, hence $C = C_1 \cup \ldots \cup C_n$ is the union of disjoint Jordan curves C_j, $j = 1, \ldots, n$ with pairwise disjoint interiors. We set $\text{ext}(C) = \text{ext}(C_1) \cap \cdots \cap \text{ext}(C_n)$ and $\text{int}(C) = \text{int}(C_1) \cup \cdots \cup \text{int}(C_n)$. In this case we set $\rho_C := \rho_\Omega$.

In the next theorem we need the function $f(\rho)$ given in Lemma 5.1 (*height* of the *catenoid* $\mathcal{M}_\rho$ arising orthogonally from the slice along a circle of radius ρ_Ω).

THEOREM 5.1. *Let Ω be an admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function taking zero boundary value data on $\partial\Omega$, everywhere continuous except maybe at a finite set $S \subset \partial_\infty\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^2 \times \mathbb{R}$ be the union of the graph of g restricted to $\partial_\infty\Omega$ with the vertical segments at the points of $\partial_\infty\mathbb{H}^2$ of discontinuities of g.*
If the height function t of Γ_∞ satisfies $-f(\rho_\Omega) \leqslant t \leqslant f(\rho_\Omega)$, then there exists a minimal vertical graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.

Particularly, if $C \subset \mathbb{H}^2 \times \{0\}$ is a Jordan curve satisfying the Exterior circle of radius ρ condition *and if $g : \partial_\infty\mathbb{H}^2 \to \mathbb{R}$ is a continuous function satisfying $-f(\rho_\Omega) \leqslant g(p) \leqslant f(\rho_\Omega)$ at any point $p \in \partial_\infty\mathbb{H}^2$, then there exists a unique minimal vertical graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.*

Finally, there is no such minimal graph, if $\partial\Omega$ is compact and the height function t of Γ_∞ satisfies $|t| > \pi/2$.

Proof. Consider the family of catenoids $\mathcal{M}_\rho$ given by Lemma 5.1. Notice that our assumptions imply that at each point $p \in C$ there exists a circle $\mathcal{C}_{\rho_C}(p)$ of radius ρ_C contained in $\overline{\text{int}(C)}$ with $p \in C_\rho(p) \cap C$. Let $\mathcal{M}^+_{\rho_C}(p)$ and $\mathcal{M}^-_{\rho_C}(p)$ be the the upper and lower half- catenoids cutting orthogonally the slice $t = 0$ along the circle $\mathcal{C}_{\rho_C}(p)$.

Take one of these lower half-catenoids as a subsolution, and take one of these upper half-catenoids as the supersolution ϕ in Perron process, Theorem 4.1. It follows that this family of half- catenoids provide also

a family of barriers at each point of C to our problem in the sense of
Definition 2-(2). Therefore our Dirichlet Problem (P), see Definition 1,
can be solved using Corollary 4.1.

If $C \subset \mathbb{H}^2 \times \{0\}$ is a Jordan curve satisfying the *Exterior circle
of radius ρ condition*, and if $g : \partial_\infty \mathbb{H}^2 \to \mathbb{R}$ is continuous, then the
uniqueness follows from the classical maximum principle Theorem 3.1.
This proves the first assertion of the statement.

To prove the nonexistence part assume by contradiction that there
exists a solution u such that the height function t of Γ_∞ satisfies $t >
\pi/2$. Notice that the graph of u is above the slice $t = 0$. Now choose a
catenoid $\mathcal{M}_\rho$ with t axis and large "neck" (ρ big enough) disjoint from
the graph of u. Let $\mathcal{M}_\rho(\epsilon) = \mathcal{M}_\rho + \epsilon$ be the ϵ-vertical translation of $\mathcal{M}_\rho$,
with $\epsilon > 0$ small enough. Now shrink the catenoid $\mathcal{M}_\rho(\epsilon)$ in the family
of catenoids with the same axis making the "neck" going to zero. We
will find a first interior point of contact of the graph of u with one of
these catenoids. This gives a contradiction by the maximum principle
and completes the proof of the Theorem.

$\square$

REMARK 5. A computation shows that any catenoid in $\mathbb{H}^2 \times \mathbb{R}$ has
finite total extrinsic curvature. We set here a question: is it true that
the same holds for any exterior minimal graph in $\mathbb{H}^2 \times \mathbb{R}$?

We now restrict our attention to certain admissible domains such
that each component of the boundary has two points at its asymp-
totic boundary and has at each point of its finite boundary an exterior
equidistant curve. To be more precise:

DEFINITION 4 (E-admissible unbounded domains in $\mathbb{H}^2$). Let Ω be an
unbounded domain in the slice $\mathbb{H}^2 \times \{0\}$ and let $\partial\Omega$ be its boundary.
We say that Ω is an *E-admissible domain* if

(1) Each connected component C_0 of $\partial\Omega$ is a properly embedded
    curve such that the asymptotic boundary is constituted of two
    distinct points.
(2) We require that there exists $r > 0$ such that each point of $\partial\Omega$
    satisfies the *Exterior equidistant curve of (uniform) curvature*
    $\tanh r$ *condition*; that is, at any point $p \in \partial\Omega$ there exists an
    equidistant curve E_r of curvature $\tanh r$ (with respect to the
    exterior unit normal to Ω at p), with $p \in \partial\Omega \cap E_r$ and $E_r \cap \Omega =
    \emptyset$.

Thus every E-admissible domain is an admissible domain.

If Ω is a convex domain satisfying the condition (1) of Definition 4
then Ω is an E-admissible domain.

If each connected component C_0 of $\partial\Omega$ is an equidistant curve then Ω is an E-admissible (maybe nonconvex) domain.

If Ω is an unbounded E-admissible domain then we denote by $r_\Omega \geqslant 0$ the infimum of the set of these r. If Ω is a convex E-admissible domain then $r_\Omega = 0$.

We will use in the next result the function H defined by Formula (1) in Proposition 2.1.

THEOREM 5.2. *Let Ω be an E-admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function taking zero boundary value data on $\partial\Omega$, everywhere continuous except maybe at a finite set $S \subset \partial_\infty\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^2 \times \mathbb{R}$ be the union of the graph of g restricted to $\partial_\infty\Omega$ with the vertical segments at the points of discontinuities of g. If the height function t of Γ_∞ satisfies $-H(\cosh r_\Omega) \leqslant t \leqslant H(\cosh r_\Omega)$, then there exists a minimal vertical graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.*

Proof. The proof is the same as in Theorem 5.1, replacing the minimal catenoids by the minimal surfaces invariant by hyperbolic translations M_d, $d > 1$, given in Proposition 2.1. This completes the proof of the Theorem. $\square$

Notice that if Ω is convex then it is E-admissible and $r_\Omega = 0$, thus $H(\cosh r_\Omega) = \infty$.

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